THE QUATERNARY SQUARED DESIGN TABLE IN EXPERIMENTAL DESIGN

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Abstract: Two-level fractional factorial experiments have been used extensively in many areas such as quality engineering, agriculture, pharmaceutical chemistry, management, etc. Geometrical designs proposed by Plackett & Burman (1946) are two-level fractional factorial designs. Using a number representation system whose base is a power of 2, Tsai (1999) developed an easy algorithm for two-factor interaction columns in geometrical designs. Based on the results of base 4, in this article we propose a Quaternary Squared Design Table (QSDT) which is useful in planning an experiment. A QSDT has a nice structure of confounding relationships so that users could identify any interaction columns in a straightforward manner without looking up tables. Some special designs constructed by the QSDT are also discussed.

Key words: Fractional factorial design, Geometrical design, Quaternary squared design table.

1. INTRODUCTION

Two-level fractional factorial experiments have been used extensively in many areas such as quality engineering, agriculture, psychology, pharmaceutical and chemical industries, material science, animal science, management, etc. How to choose a "good" design becomes an important issue. There are some well known criteria for evaluating goodness of a design. A 2^{k-p} fractional factorial design with k factors of 2 levels and $n=2^q$ runs, where q=k-p, is uniquely determined by p independent defining words. A word consists of letters which are names of factors denoted by A, B, … (or 1,2,…). The number of letters in a word is called word-length, and the group formed by the k defining words is the defining contrast subgroup. The vector $W=(A_3,\ldots,A_k)$ is called the word-length pattern, where A_i denotes the number of words of length i in the defining contrast subgroup. The idea of resolution proposed by Box and Hunter (1961a,b), is defined as the smallest r such that $A_r \geqq 1$. Fries and Hunter (1980) proposed the minimum aberration (MA) criterion to evaluate designs. For any two designs d_1 and d_2 with r being the smallest value such that $A_r(d_1) \neq A_r(d_2)$, d_1 has less aberration than d_2 if $A_r(d_1) < A_r(d_2)$. If there is no design with less aberration than d_1, then d_1 has minimum aberration. Fractional factorial designs are called "even" if their defining contrast subgroup consists entirely of even-length words. In Wu and Chen (1992), any two-factor interaction (2fi) that is not aliased with any main effect or other 2fi's is called "clear".

Many articles have provided some tools for planning experiments, for examples, Kacker and Tsui (1990) proposed interaction graphs as graphical aids for planning two-level fractional factorial experiments; Wu and Chen (1992) proposed a graph-aided method for planning two-level experiments when certain interactions are important; Chen, Sun and Wu (1993) provided a catalogue of two-level and three-level fractional designs with small runs.

Geometrical designs proposed by Plackett & Burman (1946) are two-level fractional factorial designs and they can be constructed easily by a successive doubling method. Using a number representation system whose base is a power of 2, Tsai (1999) developed an easy algorithm for 2fi columns in geometrical designs. Based on the results of base 4, this article is to propose a useful tool, called Quaternary Squared Design Table (QSDT), for planning two-level fractional factorial experiments. Geometrical designs and the number representation algorithm for 2fi are stated first, then a basic Quaternary Squared Design Table and its extension are proposed, finally some special designs using QSDT are discussed.

2. GEOMETRICAL DESIGNS

A doubling method was given by Plackett and Burman (1946, p313): If A is orthogonal, $\mathbf{B}=\begin{bmatrix}\mathbf{A} & \mathbf{A}\\ \mathbf{A} & -\mathbf{A}\end{bmatrix}$ is also orthogonal and has double the order of A. Note that B can be expressed as $B=[A_L, A_R]$, where $A_L=[A, A]'$, $A_R=[A, -A]'$, and A_R is called "fold-over" by Box and Wilson (1951). Starting from G_2, geometrical designs (GD's) can be obtained by the successive doubling method as follows:

$$\mathbf{G}_2=\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix},\quad \mathbf{G}_4=\begin{bmatrix}\mathbf{G}_2 & \mathbf{G}_2\\ \mathbf{G}_2 & -\mathbf{G}_2\end{bmatrix}=\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & -1 & 1 & -1\\ 1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1\end{bmatrix},\ \text{and}\ \ \mathbf{G}_8=\begin{bmatrix}\mathbf{G}_4 & \mathbf{G}_4\\ \mathbf{G}_4 & -\mathbf{G}_4\end{bmatrix}=\begin{bmatrix}1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1\\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1\\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1\\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1\\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1\\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1\end{bmatrix}.$$

Similarly, all geometrical designs with higher order such as G_{16}, G_{32}, G_{64}, etc. could be obtained easily by using "Excel" functions such as "copy" and "replace" without any computation, or even by hand writing.

3. NR ALGORITHM FOR 2FI IN GD’S

The doubling method has a nice recursive property for obtaining two-factor interactions (2fi’s). By observing the basic matrix of doubling $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, the 2fi of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, respectively. This property is well preserved during the process of doubling. Now, if a geometrical design is partitioned as left-half and right-half parts such as $G_{2n}=[A_L, A_R]$, where $A_L=[G_n, G_n]'$, $A_R=[G_n, -G_n]'$, then three confounding relationships can be observed as follows:

(1) The 2fi column of any two given columns in A_L remains in A_L.
(2) The 2fi column of any two given columns in A_R remains in A_L.
(3) The 2fi column of any two given columns of which one is in A_L and the other one is in A_R remains in A_R.

Namely, the basic rule for 2fi is “left by left → left”; “right by right → left”; “left by right →right”, or in short $A_L \times A_L \rightarrow A_L$; $A_R \times A_R \rightarrow A_L$; $A_L \times A_R \rightarrow A_R$. This rule can be easily extended to higher order interactions.

Based on a number representation system whose base is a power of 2, Tsai (1999) proposed an efficient method, called NR method, for obtaining the 2fi column. Let E denote the 2fi column of two given columns C and D, namely, $E=T_n(C,D)$, then the NR method for quaternary case is stated as below:

1. Convert two given column numbers C and D into digits with base 4.
2. Compute $E=T_n(C,D)$ digit by digit according to the basic confounding relationships in G_4 such as $T_4(1,2)=3$; $T_4(1,3)=2$; $T_4(2,3)=1$; $T_4(0,j)=j$; $T_4(j,j)=0$, j=0,1,2,3.
3. Convert the resulting digit numbers with base 4 back to a decimal digit number.

Two examples are given below:

a. $T_{16}(9,14)=T_{16}([21,32])=[13]=7$;
b. $T_{128}(13,99)=T_{128}([0031],[1203])=[1232]=110$.

4. QUATERNARY SQUARE DESIGN TABLE

Based on the above NR algorithm for the quaternary case, we can easily construct a basic “Quaternary Squared Design Table (QSDT)” as in Table 1.

00	10	20	30
01	11	21	31
02	12	22	32
03	13	23	33

Table 1: A basic quaternary squared design table (QSDT) for G_{16}.

This is a two-level design with 16 runs; all columns from 0 to 15 are presented in a quaternary system and arranged in a 4x4 square table; column 0 is an identity column which can not be assigned any factor. Applying the basic confounding relationships in G_4, we can obtain the 2fi of any two given columns in a straightforward manner without looking up any interaction table. For example, the 2fi of column 9 (21) and column 14 (32) is column 7 (13). When the run size of a design increases, the above basic table could be double, triple, etc. For example, three digits are required for a 32-run two-level design to represent columns 0 to 31, and two QSDT’s are put together as in Table 2 & 3.

000	010	020	030	100	110	120	130
001	011	021	031	101	111	121	131
002	012	022	032	102	112	122	132
003	013	023	033	103	113	123	133

Table 2: Two joint quaternary squared design tables for G_{32}.

0				1			
00	10	20	30	00	10	20	30
01	11	21	31	01	11	21	31
02	12	22	32	02	12	22	32
03	13	23	33	03	13	23	33

Table 3: Alternative two joint quaternary squared design tables for G_{32}.

The left-most digit was shown on the top of the table since the other two digits are all the same for two basic tables. Clearly, the left-half part contains column 0 to 15 and the right-half part contains columns 16 to 31. Note that (a) the 2fi of any two columns from the right-half part remains in the left-half part; (b) the 2fic of any two column from the left-half part remains the left-half part; (c) the 2fic of any two columns of which one is from the left-half part and the other one is from the right-half part remains in the right-half part. For another example, three digits are required for a 64-run two-level design to present columns 0 to 63, and four basic tables are put together as in Table 4:

0				1				2				3			
00	10	20	30	00	10	20	30	00	10	20	30	00	10	20	30
01	11	21	31	01	11	21	31	01	11	21	31	01	11	21	31
02	12	22	32	02	12	22	32	02	12	22	32	02	12	22	32
03	13	23	33	03	13	23	33	03	13	23	33	03	13	23	33

Table 4: Four joint quaternary squared design tables for G_{64}.

Let the value of left-most digit i denote the i^{th} basic table as QSDT-I, for example QSDT-0 contains column 0 to 15. Since the basic confounding relationships in G_4 are applied to each digit independently, QSDT-0 to QSDT-3 also follow the same confounding relationships, for example, the 2fi of any two columns from QSDT-3 remains in QSDT-0, the 2fi of any two columns of which one is from QSDT-1 and the other one is from QSDT-2 remains in QSDT-3. These quaternary square design tables are very easy to implement using a spreadsheet such as "Excel" even for the case of large runs, and are visually appealing for uses to plan a better design.

5. SOME SPECIAL DESIGNS

Chen, Sun and Wu (1993) provided a catalogue of two-level and three-level fractional factorial designs with small runs, the additional column numbers other than independent columns are reported and the design numbers are arranged as [k-p.i] in the given tables. Two special designs will be adopted to illustrate the use of a QSDT, and how to construct other alternative designs. For a QSDT with n=8, the left-half part contains columns 0-4 while the right-half part contains 4-7; for a QSDT with n=16, the left-half part contains columns 0-7 while the right-half part contains 8-15; similarly for any size of n.

(a) Even MA Resolution IV designs

The series of $2^{4-1}_{\mathbf{IV}}$, $2^{8-3}_{\mathbf{IV}}$, $2^{16-11}_{\mathbf{IV}}$, $2^{32-26}_{\mathbf{IV}}$,... has twice as many runs (n=2k) as factors (k) and are even MA resolution IV designs, in which each of k-1 alias sets contains k/2 2fi's. Table 5 shows an example of k=8 in [8-4.1], factors are assigned to columns {1,2,4,5,8,11,13,14}, each of 7 blank columns contains 4 2fi's, and W=(0,14,0,0,0,1). Table 6 shows another similar even MA resolution design in which factors are assigned to columns 8-15 in the right-half part, each of 7 blank columns in the left-half part contains 4 2fi's, and with the same W=(0,14,0,0,0,1).

00	10	20	
01			31
02			32
	13	23	

Table 5: The design of 2^{8-4} with even MA resolution IV in [8-4.1].

00		20	30
		21	31
		22	32
		23	33

Table 6: The similar design of 2^{8-4} with even MA resolution IV.

Furthermore, Table 7 shows an example of k=16 in [16-11.1], factor are assigned to columns {1,2,4,7,8,11,13,14,16,19,21,22,25,26,28,31}, each of 15 blank columns contains 8 2fi's, and W=(0,140,0,448,…). Table 8 shows another similar even MA resolution design in which factors are assigned to columns 16-31 in the right-half part, each of 15 blank columns in the left-half part contains 8 2fi's, and with the same W=(0,140,0,448,…). Moreover, all three-factor interactions are in the right-half part; all four-factor interactions are in the left-half part, namely, odd-factor interactions are in the right-half part and even-factor interactions are in the left-half part. This fact shows the nice confounding relationships in a QSDT which is visually appealing.

000	010	020		100			130
001			031		111	121	
002			032		112	122	
	013	023		103			133

Table 7: The design of 2^{16-11} with even MA resolution IV in [16-11.1].

000				100	110	120	130
				101	111	121	131
				102	112	122	132
				103	113	123	133

Table 8: The similar design of 2^{16-11} with even MA resolution IV.

Although [8-4.1], [16-11.1] and [32-26.1] reveal some special patterns in a QSDT, larger sizes are more complicated and not available in Chen, Sun and Wu (1993). In summary, all factors are assigned to the right-half part of a QSDT can easily generate an even MA resolution IV design. This work for any case of large size too, for example, all factors being assigned to columns 32-63 in a QSDT is an even MA resolution IV design for the case of $2_{\mathbf{IV}}^{32-26}$.

(b) Maximal number of Clear 2fi's

Table 9 shows that factors are assigned to columns {1,2,3,4,5,6,7,8} in [8-4.6], the 2fi of any two factors in left-half part remains in left-half part; the 2fi of column 8 with any column in left-half part remains in right-half part. Obviously, seven clear 2fi's in columns 9-15 are clear. If column 8 is replaced by any one from columns 9-15 in Table 10, then the design also have seven clear 2fi's in right-half part. Following the same structure, it is easy to extend the result to a larger size, for example, Table 11 shows a similar design for 2^{16-11} in which 15 factors are assigned to left-half part, one factor is to right-half part, and 15 clear 2fi's are in right-half part. Note that Chen, Sun and Wu (1993) did not provide such designs.

00	10	20	
01	11		
02	12		
03	13		

Table 9: The design of 2^{8-4} with seven clear 2fi's in [8-4.6].

00	10		30
01	11		
02	12		
03	13		

Table 10: The similar design of 2^{8-4} with seven clear 2fi's.

000	010	020	030	100	110	120	130
001	011	021	031	101	111	121	131
002	012	022	032	102	112	122	132
003	013	023	033	103	113	123	133

Table 11: The design of 2^{16-11} with fifteen clear 2fi's.

CONCLUSIONS

In this article a quaternary squared design table (QSDT) was proposed as a new tool for an experimenter to plan a better design. A QSDT is visually appealing and is easy to use for a large size. A QSDT has a nice structure of confounding relationships so that users could identify any interaction columns in a straightforward manner without looking up tables. Some special designs such as even minimum aberration resolution IV or maximal clear 2fi's could be constructed easily with the help of a QSDT, and this can be extended to the case of larger size without any difficulty. Since the QSDT is a new tool for a two-level fractional factorial design, some properties deserves more studies, for example, how to find the numbers of length-three or length-four words in its defining contrast subgroup.

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