ESTIMATING REORDER POINTS OF CORRELATED DEMANDS BY SOME NONPARAMETRIC METHODS

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ABSTRACT

When the daily demands are modeled by a general autoregressive and moving average (ARMA) process and the lead times are allowed to vary randomly, the moments of the lead-time demand (L) are derived. Based on these moments, the reorder points (*ROP*s) at a given service level are estimated under a continuous review inventory policy by procedures such as the Pearson system, normal approximation, Charlier series, Monte Carlo method and bootstrap. The Charlier series and bootstrap procedure are most satisfactory among the different methods with sample realizations. We further demonstrate that an adverse effect on the quality of the estimated *ROP*s may occur when the correlated nature of the demand process is ignored.

Keywords: Lead-Time Demand; Pearson System; Charlier Series; Monte Carlo Method; Bootstrap

INTRODUCTION

The two-staged random process $L = \sum_{t=1}^{T} Y_t$, where both T and Y_t are random, is useful for describing many business and industrial situations. In a continuous review inventory system, let Y_t be the daily demand (or other unit time demand) at time period t, and let T be the stochastic lead time, then L is the lead-time demand of the inventory item. The knowledge of the distribution of L is an integral part for many basic inventory decisions such as determining the reorder points (*ROP*s), estimating order quantity and finding the expected-lost-sales when both Y_t and T follow some stochastic processes.

An optimal solution of determining simultaneously the *ROP*s and the order quantity may be obtained by minimizing the total expected inventory cost. See, for example, the procedure suggested by Hadley and Whitin [1], and Silver and Peterson [2]. Rather than minimizing the inventory cost, this paper attempts to find solution of the *ROP*s by satisfying the customers at a pre-determined service level. This approach of estimating the *ROP*s has been widely used. See, for example, Ray [3][4][5], Bagchi, Hayya and Ord [6], Lau and Wang [7], Fotopoulos, Wang and Rao [8], and An, Fotopulos and Wang [9] to name a few.

Various approaches are available for estimating the distribution of L. One such approach assumes that T and Y_t are mutually independent and that the successive realizations of Y_t are *i.i.d*. A compound distribution of L may be derived. See, for example, Das [10][11], Burgin and Norman [12], Bott [13], Tadikamlla [14], Ord and Bagchi [15], Bagchi, Hayya and Ord [6], and Bagchi [16]. However, Y_t may very well be correlated over time in many demand situations such as discussed in Ray [3] and Badinelli [17]. The compound distribution approach is therefore not suitable for a correlated demand structure. A significant improvement from that is the curve-fitting procedure based on the moments of L. It requires only the existence of the moments of T and Y_t without any parametric assumptions for the components nor the *i.i.d*. assumption on the $Y_t^{'}$s. See, for example, Kottas and Lau [18], Ray [3][4][5], Lau and Zaki [19], Fotopoulos, Wang and Rao [8], and An, Fotopoulos and Wang [9].

When Y_t is correlated over time, the class of ARMA processes of Box and Jenkins [20] is considered here for modeling the inventory demand. In fact, Badinelli [17] suggested that there were several situations in manufacturing where an ARMA demand process was likely to occur.

For example, a capacity-constrained production facility that places demand on a supplier, the demand pattern seen by the supplier is autocorrelated and can be modeled by an ARMA process. In a multi-echelon inventory system such as a warehouse-retailer distribution system, demand seen by a supplier can also be modeled by an ARMA process. When an inventory demand has a memory pattern to its past or when the demand depends on the previous inventory levels, then, an ARMA process is appropriate for representing such correlated situations.

In this chapter, the exact expressions of the first four moments of *L* are derived for a general ARMA demand process with stochastic lead time. The result on the moments of *L* is more general than the previous work by Lau and Wang [7], and An, Fotopoulos and Wang [9]. The distribution of *L* and specifically the *ROP* for a given service level are then estimated by methods such as the Pearson system, normal approximation, Charlier series, Monte Carlo simulation and bootstrap procedure. When samples are available, the estimated *ROP*s show that the nonparametric methods such as the Charlier series and bootstrap procedure both are good whereas the well-known Pearson system is disappointing in some test cases.

We further consider a situation where the demand Y_t is correlated over time but was mistreated as *i.i.d.* Significant effects on the estimated *ROP* and service levels are demonstrated numerically. The economic consequences can be quite substantial.

ARMA DEMAND AND THE MOMENTS OF *L*

Suppose that the inventory demand has Y_t has memory pattern of its past and its past and is modeled by the Box and Jenkins' [20] stationary ARMA(p,q) process

$$\Phi(\mathrm{B})Y_t = a + \Theta(\mathrm{B})\varepsilon_t \tag{1}$$

Where *a* is a constant, ε_t is a sequence of random errors with mean 0 and variance σ_ε^2. $\Phi(\mathrm{B})$ and $\Theta(\mathrm{B})$ are polynomials, where $\Phi(\mathrm{B}) = 1 - \sum_{i=1}^{p} \phi_i B^i$, $\Theta(B) = 1 - \sum_{j=1}^{q} \phi_j B^j$, and *B* is the backshift operator such that $B^m Y_t = Y_{t-m}$. It is assumed that the roots of the polynomial equations, $\Phi(\mathrm{B}) = 0$ and $\Theta(\mathrm{B}) = 0$ lie outside the unit circle so that Y_t is stationary as well as invertible. Observe that the coefficients $\phi_1, \phi_2, \cdots$ are the weights that relate the previous inventory demands to the current demand; the coefficients $\theta_1, \theta_2, \cdots$ are the weights that relate the previous random errors to the current demand. With the class of correlated demand processes just defined, the resulting inventory decisions will certainly be more complex than the traditional approach. However, when samples of the inventory demand are available, the Box and Jenkins [20] iterative modeling procedure offers effective ways for model identification, parameter estimation, diagnostic checking, validating and forecasting. Furthermore, some standard software packages such as SAS, SPSSX, MINITAB, SCA, etc. all have these time series routines, which allow for practical implementations. Since our main focus in this paper is the inventory decision, we are bypassing the iterative modeling procedure just mentioned, and instead, assuming that the ARMA model in (1) for an inventory demand is verifiable and appropriate.

For the time origin at zero, let there be at least s=p + q previous demands $\tilde{\mathbf{Y}} = (\tilde{Y}_0, \tilde{Y}_{-1}, \cdots, \tilde{Y}_{1-s})$. The observed residual errors $\tilde{\varepsilon}_{-i}$ (i = 0,1,...) are defined as

$$\tilde{\varepsilon}_{-i} = \tilde{Y}_{-i} - \hat{Y}_{-i} \qquad (2)$$

where $\hat{Y}_{-i}$ is a one-step ahead forecasted value of Y_{-i} from (1) based on the initial demands $\tilde{\mathbf{Y}}$. Note that $\tilde{\mathbf{Y}}$ and $\tilde{\varepsilon}_{-i}$ are independent of the lead time T and the *i.i.d.* sequence $\{\varepsilon_t, t \in N\}$, where N is the set of positive integers. It follows from the initial conditions on $\tilde{\mathbf{Y}}$ and (2) that (1) can be expressed recursively as a weighted linear combination of the random disturbances ε_t

$$Y_t = a_t + \sum_{j=0}^{t-1} \psi_j \varepsilon_{t-j} \ , \qquad (3)$$

where $\psi_j (j = 0, 1, \cdots)$ are the weights which related the past random errors to the current demand. Mathematically, ψ_j can be computed from $\psi(\mathbf{B}) = \Theta(\mathbf{B}) / \Phi(\mathbf{B}) = \sum_{j=1}^{\infty} \psi_j B^j$ and a_t has an autoregressive relation

$$a_t = \sum_{i=1}^{p} \phi_i a_{t-i} + a - \sum_{j=t}^{q} \theta_j \tilde{\varepsilon}_{t-j} \qquad (4)$$

with initial values $a_{-i} = \tilde{Y}_{-i}$ for i = 0, 1, 2,$\cdots$,(p-1). Although a_t is an infinite sum which is written as $a_t = a \big/ \Phi(\mathbf{B}) + \sum_{j=t}^{\infty} \psi_j \tilde{\varepsilon}_{t-j}$, the value of a_t, however, can be evaluated with only a finite number of observed residuals as stated in (4). The expression of Y_t by (3) is therefore computationally convenient. Solving the difference equation of (4) with the given initial values, we have

$$a_t = \frac{a}{\Phi(\mathrm{B})} + \sum_{i=1}^{p} c_i S_i^{t-1} = \mu_Y + \sum_{i=1}^{p} c_i S_i^{t-1}, \qquad t=1,2,\cdots, \tag{5}$$

where μ_Y is the unconditional mean of Y_t, S_i^{-1} are the roots of polynomial $\Phi(\mathrm{B}) = 0$ and c_i are determined by the initial values as well as the parameters of the ARMA process. The lead-time demand L is then given by

$$L = \sum_{t=1}^{T} Y_t = \sum_{t=1}^{T} a_t + \sum_{t=1}^{T} b_t \varepsilon_{T+1-t} = \mu_Y T + \sum_{i=1}^{p} d_i (S_i^T - 1) + \sum_{t=1}^{T} b_t \varepsilon_{T+1-t}, \tag{6}$$

where

$$b_t = \sum_{j=1}^{t-1} \psi_j \text{ and } d_i = -c_i / (1 - S_i). \tag{7}$$

Setting the notations as follow:

$$\mu_\varepsilon = E[\varepsilon_t] = 0,\ \ \mu_T = E[T],\ \mu_L = E[L \mid \mathbf{Y}],\ \mu_k(\varepsilon) = E[(\varepsilon_t - \mu_\varepsilon)^k],$$

$$\mu_k(T) = E[(T - \mu_T)^k], \quad \mu_k(L) = E[(L - \mu_L)^k \mid \mathbf{Y}], \text{ for } k = 2,3,4.$$

The first four moments of L can be derived based on a set of relatively mild conditions. We have the following theorem.

Theorem: Suppose that the demand Y_t is generated by the ARMA (p,q) process of (3). If (A1): the lead time T and the random error ε_t are mutually independent; (A2): the ARMA (p,q) demand process of (1) is stationary; (A3): the probability generating function of T and the fourth moments of ε_t both exist, then, the first four moments of L, defined by (6), are given by (8), (11), (12) and (13).

Proof it is easy to see from (6) that the mean of L is given by

$$\mu_L = E[\sum_{t=1}^{T} Y_t] = \mu_Y \mu_T + \sum_{i=1}^{p} d_i (E[S_i^T] - 1). \tag{8}$$

Hence, the deviation

$$L - \mu_L = D(T) + \sum_{t=1}^{T} b_i \varepsilon_{T+1-t}, \tag{9}$$

where

$$D(T) = \mu_Y(T - \mu_Y) + \sum_{i=1}^{p} d_i(S_i^T - E[S_i^T]) . \qquad (10)$$

Let $G_T(m,n) = D^m(T)\Sigma_{t=1}^{T} b_t^n$ for nonnegative integers *m* and n. The second, third and fourth moments of *L* under assumptions (A1)-(A3), are obtained by

$$\mu_2(L) = E[D^2(T)] + \mu_2(\varepsilon)E[G_T(0,2)] . \qquad (11)$$

$$\mu_3(L) = E[D^3(T)] + 3\mu_2(\varepsilon)E[G_T(1,2)] + \mu_3(\varepsilon)E[G_T(0,3)] . \qquad (12)$$

$$\mu_4(L) = E[D^4(T)] + 6\mu_2(\varepsilon)E[G_T(2,2)] + 4\mu_3(\varepsilon)E[G_T(1,3)]$$

$$+\mu_4(\varepsilon)E[G_T(0,4)] + 3\mu_2^2(\varepsilon)E[G_T^2(0,2) - G_T(0,4)] . \qquad (13)$$

The expectations above can be evaluated by

$$E[G_T(m,n)] = \sum_{t=1}^{\infty} G_t(m,n) \cdot P_r(T = t) . \qquad \textit{Q.E.D.}$$

Although the derivation of the moments in the Theorem is straightforward, the moments obtained by the Theorem can be regarded as a basis for estimating *L* and *ROP*s. From (10) we note that *D(T)* represents the deviation of the conditional mean of *L* (for given time *T*) from the overall mean μ_L. Experience shows that the variation of *L* of (9) is usually dominated by the variation of *D(T)*. In fact, *D(T)* includes the linear term of *T*, which makes it possible to investigate the effect of the variation of *T* on *L*.

What happens if a true demand process is correlated but was mistreated as *i.i.d.*? What kind of error may occur? To examine that situation more specifically, suppose that Y_t is expressed as

$$Y_t = \mu_Y + \nu_t ,$$

where ν_t is regarded as *i.i.d*. but is actually correlated over time. Then *L* is expressed by

$$L = \sum_{t=1}^{T} Y_t = \mu_Y T + \sum_{t=1}^{T} \nu_t \ .$$

Note that $\nu_t = \Sigma_{j=0}^{\infty} \psi_j \varepsilon_{t-j}$ where ψ_j and ε_{t-j} are defined by (3). Then the variance of *L* is simply

$$\mu_2(L) = (\mu_Y)^2 \mu_2(T) + \mu_T \mu_2(\nu) = (\mu_Y)^2 \mu_2(T) + \mu_T \mu_2(\varepsilon) \sum_{j=1}^{\infty} \psi_j^2 \ . \quad (14)$$

where μ_Y is the unconditional mean and $\mu_2(\nu)$ is the variance of ν_t. Here $\mu_2(\nu) = \sigma_Y^2$. Comparing (14) with (11), we see tat a large error may occur in $\mu_2(L)$ when either $\mu_2(\varepsilon)$, μ_T or ψ_j are large. Errors in other moments as well as the estimated ROPs will surely arise when the ψ_j weights or the correlation structure are ignored. When this happens, numerical investigation in this study shows that substantial errors occur in the estimated ROPs and service levels in some cases.

METHODS OF ESTIMATING THE ROPs

Because of the nature of the sample realizations, different procedures for estimating the *ROP*s should be considered. If the samples on *T* are not available, then a distributional assumption on *T* must be made. The observed sequence of residuals $\{\tilde{\varepsilon}_t\}$ can be computed from the fitted ARMA model, the observed Y_t and the assumed distribution of *T*. Estimate of *D(T)* of (10) can also be obtained. Combining these, the estimated first four moments of *L* can be derived according to the Theorem. Estimating the ROPs by the Pearson system, normal approximation and Charlier series can then be available in a straightforward manner. Computationally, the pseudo random numbers of *L* and the empirical distribution *L* may be obtained form the residual sequence $\{\tilde{\varepsilon}_t\}$ and the assumed distribution of *T*. The *ROP*s can then be estimated via the Monte Carlo method and Bootstrap procedure. If the sample realization of Y_t and *T* are both available, we then first identify the ARMA process for Y_t and the distribution of *T*, estimate the parameters involved, and follow the same procedure as stated above to estimate the *ROP*s.

Pearson System

Approximating formulas for estimating the percentage points of the Pearson curves are obtained by Bowman and Shenton [21] and Soloman and Stephens [22]. Let $q_P(p)$ be the estimated *p*th percentile of the distribution of $Z = (L - \mu_L)\big/\sqrt{\mu_2(L)}$ based on the estimated moments and determined by Bowman-Shenton formulas, the ROP at *p* % service level is given by

$$ROP_P(p) = \hat{\mu}_L + q_P(p)\sqrt{\hat{\mu}_2(L)} \tag{15}$$

where $\hat{\mu}_L$ and $\hat{\mu}_2(L)$ are the estimated mean and variance of L , respectively.

Normal Approximation and Charlier Series

For a normal approximation, the estimated ROP is given by

$$ROP_N(p) = \hat{\mu}_L + z(p)\sqrt{\hat{\mu}_2(L)}\,, \tag{16}$$

where *z(p)* is the *p*th percentile of the standard normal distribution. Since *L* is usually asymmetrical, therefore, it is anticipated that the normal approximation provides a less accurate estimate.

An improvement over the normal approximation leads naturally to the Charlier series. (See for example, Kendall and Stuart [23, pp. 166-169] where the of *c.d.f* . of *L* is expressed as a series of normal *c.d.f's* as follows:

$$F_Z(z) \approx \Phi(z) - \frac{\alpha_1}{6}\Phi^{(3)}(z) + \frac{\beta_2 - 3}{24}\Phi^{(4)}(z) \tag{17}$$

where $\Phi^{(r)}(\cdot)$ is the *r*th derivative of the standard normal *c.d.f.,* and α_1 and β_2 denote the skewness and kurtoisis of *L*, respectively. Let $q_C(p)$ be the *p*th sample percentile of $F_Z(\cdot)$ according to the Charlier series when the sample values of α_1 and β_2 are used in (17). The estimated ROP by the Charlier series is then written by

$$ROP_C(p) = \hat{\mu}_L + q_C(p)\sqrt{\hat{\mu}_2(L)}\ . \qquad (18)$$

Monte Carlo Method

Based on the residual sequence $\{\hat{\varepsilon}_t\}$ and the distribution of T with either given or estimated parameters, it is possible then to generate Y_t from an estimated ARMA demand model and consequently, the generated L.

Suppose that an ARMA model is properly chosen or estimated the residual sequence $\{\hat{\varepsilon}_t\}$ can be regarded as a sequence of the realized *i.i.d.* random variables. A random number of lead times, say t_1 is generated according to the distribution of *T*. Then a random sample of size t_1, denoted by $\hat{\varepsilon}' = \{\hat{\varepsilon}_1', \cdots, \hat{\varepsilon}_{t1}'\}$ can be taken from the sequence $\{\hat{\varepsilon}_t\}$. Therefore, the Monte Carlo daily demand Y_i' can be obtained form the estimated values $\hat{a}, \hat{\phi}_1, \hat{\phi}_2$ and initial condition $\mathbf{Y}$. Consequently, the generated $L_{t1}' = \Sigma_{t=1}^{t1} Y_i'$. Alternatively, L_{t1}' can be obtained directly form the residual sequence where

$$L_{t1}' = \sum_{t=1}^{t1} \hat{a}_t + \sum_{t=1}^{t1} \hat{b}_t \hat{\varepsilon}_{t1+1-t}.$$

Suppose that a sample of n lead times is available. Based on these lead times, we independently performed m replications of the realized residuals $\{\hat{\varepsilon}_t\}$, which will produce the generated samples of the lead-time demand $L' = (L_{t1}', \cdots, L_{tm}')$ (in the numerical study, *m* is taken as 200). The empirical *c.d.f.* of L' can then be defined as follows:

$$\hat{F}_m(x) = \begin{cases} 0 & x < L_{(1)} \\ \dfrac{j}{m} & L_{(j)} \le x < L_{(j+1)} \quad j = 1, \cdots, m-1 \\ 1 & L_{(m)} \le x \end{cases}, \qquad (19)$$

where $L_{(j)}$ $(j = 1, \cdots, m)$ is the *j*th order statistic of the generated sample lead-time demands, $L^{'}$. Let $\hat{\hat{\xi}}_p$ be the *p*th sample quantile obtained from $\hat{F}_m(\cdot)$, an estimated $\hat{\hat{\xi}}_p$, as suggested by Parzen [24], is determined below:

$$\hat{\hat{\xi}}_p = m(\frac{2j+1}{2m} - p)L_{(j)} + m(p - \frac{2j-1}{2m})L_{(j+1)}, \tag{20}$$

where

$$\frac{2j-1}{2m} \le p \le \frac{2j+1}{2m} \qquad j = 1, 2, \cdots, m-1.$$

The ROP at a the *p%* service level is estimated as

$$ROP_M = \hat{\hat{\xi}}_p. \tag{21}$$

Bootstrap Procedure

Efron [25][26] proposed a resampling procedure, entitled the *bootstrap*, which can be used here for estimating the ROPs. When the $\{\hat{\varepsilon}_t\}$ and the distribution of *T* are available, the bootstrap procedure should then be applied to the residuals $\{\hat{\varepsilon}_t\}$. See for example, Freedman and Peters [27], Peters and Freedman [28] for the application to regression models; Thombs and Schucany [29] for the application to the time series models. The bootstrap procedure for estimating the ROP can be described as follows:

(i) Construct the empirical distribution for the residual sequence $\{\hat{\varepsilon}_t\}$.

(ii) For a bootstrap resample of size n^* (n^*=100 in the numerical investigation), $\hat{\varepsilon}^{'} = \{\hat{\varepsilon}_1^{'} \cdots, \hat{\varepsilon}_{n^*}^{'}\}$ is taken from the residual sequence $\{\hat{\varepsilon}_t\}$ with replacement. (To improve the estimated quantile, n^* can be taken as larger than the available sample size of the residual sequence. See, for example, Bickel and Freedman [30], Barabas, Csorgo, Horvath and Yandell [31].)

(iii) A random lead time t_1 is generated according to the distribution with either given or estimated parameter.

(iv) A random sample of size t_1 (sampling with replacement) is generated from the bootstrap resample $\hat{\varepsilon}^{'}$ to obtain $\hat{\varepsilon}^{*}=\{\hat{\varepsilon}_1^{*},\cdots,\varepsilon_{t1}^{*}\}$. This will in turn generate a sequence of daily demands $Y_1^{*},\cdots,Y_{t1}^{*}$ and thus, the generated lead-time demand L_{t1}^{*} which is given by

$$L_{t1}^{*}=\sum_{t=1}^{t1}Y_t^{*}=\sum_{t=1}^{t1}\hat{a}_t+\sum_{t=1}^{t1}\hat{b}_t\hat{\varepsilon}_{t1+1-t}^{*},$$

where $\hat{a}_t$ and $\hat{b}_t$ denote the estimates of a_t and b_t defined by (4) and (7), respectively.

(v) Independently repeat the steps (iii) and (iv) for a reasonably large number, say *m* times so as to obtain the bootstrap lead-time demands $L^{*}=(L_{t1}^{*},\cdots,L_{tm}^{*})$. The estimated *p*% quantile or the ROP^{*} is obtained by (21).

(vi) Independently repeat the steps (ii) to (v) *B* number of times, which will yield a set of estimated ROPs, $ROP^{*}(j), j=1,\cdots,B.$ The estimated bootstrap ROP is given by

$$\overline{ROP}^{*}=E_{*}(ROP)=\sum_{j=1}^{B}ROP^{*}(j)/B. \quad (22)$$

EVLUATIONS

In practice, so long as the lead times and the empirical distribution of *L* are available, the ROP can obviously be estimated by the various methods suggested. The question now is how to evaluate their performance. A simulation study is conducted by assuming Y_t and *T* to belong to some parametric distributions. The quality of the resulting estimated ROP_s is then measure by: (i) The probability coverage of the estimated ROP at, say *p* % service level, $P_r(L\leq ROP)$. The closer this probability is to the pre-determined *p%* service level, the more accurate is the proposed method. (ii) The sample standard error of ROP. The smaller is the sample standard error, the more efficient is the model.

Lead Times and Demand Structure

It happens frequently in an inventory control situation that only partial sample information are available. For example, the realizations of the daily demand Y_t are available from the past records but the records on lead time T are either totally absent or extremely limited, and one must rely upon some reasonable distribution assumption on T. See, for example, Ray [3][4][5].

(i) <u>Lead Times</u>

Similar to Bott [13], we find the mean and variance of the realized lead times wherever possible, a distribution of T may be suggested according to the following guideline;

(i) When Var/Mean >1 and Var = Mean(Mean-1), we suggest that the geometric lead time may be used

$$P_r(T=t)=\pi(1-\pi)^{t-1}, \quad t \geq 1$$

(ii) When Var/Mean = 1, we suggest that the truncated Poisson lead time may be used

$$P_r(T=t)=e^{-\mu}\frac{\mu^t}{t!}\Big/(1-e^{-\mu}), \qquad t \geq 1$$

(iii) When Var/Mean <1, the truncated binomial lead time may be use

$$P_r(T=t)=\binom{n}{t}\pi^t(1-\pi)^{n-t}/(1-(1-\pi)), \qquad t=1,2,\cdots,n.$$

(iv) In general, for a symmetric or non-symmetric distribution, on may consider the truncated beta-binomial lead time

$$P_r\,(T=t) = \binom{n}{t}\frac{\Gamma(r_1+r_2)}{\Gamma(r_1)\Gamma(r_2)}\,\frac{\Gamma(t+r_1)\Gamma(n+r_2-t)}{\Gamma(n+r_1+r_2)}\Big/(1-p_0)\,,$$

Where

$$p_0 = \frac{\Gamma(r_1+r_2)\Gamma(n+r_2)}{\Gamma(r_1)\Gamma(n+r_1+r_2)}, \qquad t = 1, 2, \cdots, n.$$

Note that

$$\frac{\text{Var}}{\text{Mean}} \approx \frac{r_1 r_2 (n+r_1+r_2)}{(r_1+r_2)(r_1+r_2+1)},$$

and the distribution is symmetrical when $r_1 = r_2$, non-symmetrical, otherwise.

(ii) Daily Demands

Suppose that the realizations of Y_t are available. One could tentatively identify an ARMA process suitable for the realization of Y_t; estimate the parameters involved; check the adequacy of the process and suggest for modification; and finally determine the model to be entertained. For the cases when the demand follows an AR(1) or MA(1) process, they have been studied previously by An, Fotopoulos and Wang [9], therefore, the realizations of Y_t in the example below are assumed to be generated from an AR(2) process, where the current demand is correlated with all previous demands in decreasing magnitude to the remote past.

Let Y_t be an AR(2) process with initial observation $\tilde{Y}_{-1} = \mu_Y = 20$, say, and $\tilde{Y}_0 = \mu_Y + k\sigma_Y$ where σ_Y stands for the standard deviation of Y_t. The process can be written as

$$\begin{aligned} Y_t &= a + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \varepsilon_t \\ &= a_t + \sum_{j=0}^{t-1} \psi_j \varepsilon_{t-j} \,. \end{aligned} \qquad (23)$$

where ε_t is distributed as $\mathrm{N}(0, \sigma_\varepsilon^2)$ with σ_ε^2 = 25, say. Thus,

$$L = \sum_{t=1}^{T} Y_t = \sum_{t=1}^{T} a_t + \sum_{t=1}^{T} b_t \varepsilon_{T+1-t} \,, \qquad (24)$$

where

$$a_t = \phi_1 a_{t-1} + \phi_2 a_{t-2} + a, \quad t = 1, 2, \cdots, \quad \text{and} \quad a_0 = \tilde{Y}_0, \quad a_{-1} = \tilde{Y}_{-1}.$$

Also

$$b_t = \sum_{j=1}^{t-1} \psi_j = \frac{1}{p-q}\left\{ \frac{p(1-p^t)}{1-p} - \frac{q(1-q^t)}{1-q} \right\}, \quad \text{where} \quad p + q = \phi_1, \; pq = -\phi_2.$$

One can show from (8), (10) and (24) that

$$\mu_L = \mu_Y \mu_T + d_1 (E[p^T] - 1) + d_2 (E[q^T] - 1),$$

$$D(T) = \mu_Y (T - \mu_T) + d_1 (p^T - E[p^T]) + d_2 (q^T - E[q^T]),$$

where

$$\mu_Y = a/(1 - \phi_1 - \phi_2), \quad d_1 = -Ap^2/(p-q)(1-p), \quad d_2 = Bq^2/(p-q)(1-q),$$

$$A = \tilde{Y}_0 - \mu_Y - q(\tilde{Y}_{-1} - \mu_Y), \quad \text{and} \quad B = \tilde{Y}_0 - \mu_Y - p(\tilde{Y}_{-1} - \mu_Y).$$

The higher order moments of *L* can then be derived according to the Theorem.

Performance Measures

(i) Probability Coverage:

Let *R* be an estimated ROP. Then, for a conditional *c.d.f.* $F_{L|T}(\cdot)$,

$$P_r(L \le R) = \sum_{t=1}^{\infty} F_{L|T}(R \mid T = t) P_r(T = t)$$

$$= \sum_{t=1}^{\infty} \Phi\left(\frac{R - \mu(t)}{\sigma(t)}\right) P_r(T = t), \tag{25}$$

where $\Phi(\cdot)$ is the standard normal *c.d.f.*, $\mu(t)$ is the conditional mean of *L* and $\sigma(t)$ is the conditional standard deviation of *L* for given *T*=*t*, when the error structure ε_t of the process is assumed to be *i.i.d*. normal $N(0,\sigma_\varepsilon^2)$. The closer is the probability coverage to the pre-determined service level of *p*%, the more accurate is the method.

(ii) Sample Standard Error

Suppose that *M* number of independent simulation trials are performed. This yields $ROP_1,\cdots,ROP_M$. In the numerical results, *M*=100. The estimated ROP, say *R*, is given by

$$R = \overline{ROP} = \sum_{i=1}^{M} ROP_i / M . \tag{26}$$

with the sample standard error of *R*,

$$SE(R) = \sqrt{\sum_{i=1}^{n} (ROP_i - \overline{ROP})^2 / M(M-1)} . \tag{27}$$

The standard error (27) is used for the Pearson system, normal approximation, Charlier series and Monte Carlo method. The bootstrap procedure involves *M* independent simulation trials where each trial contains *B* number of independent bootstrap replications (in the numerical investigation M=100 and B=100). Therefore, one obtains an estimated *ROP* for the *i*th simulation trial and the *j*th bootstrap replication, $ROP^*(i,j)$, $i=1,\cdots,M$, $j=1,\cdots,B$. The average of the *B* number of bootstrap ROPs for the *i*th simulation trial is given by

$$\overline{ROP}^*(i,\cdot) = \sum_{j=1}^{B} ROP^*(i,j)/B$$

The estimated ROP for all *M* simulation trials is then given by

$$ROP_{BOOT} = \sum_{i=1}^{M} \overline{ROP}^*(i,\cdot)/M . \tag{28}$$

and the standard error of ROP_{BOOT} for all simulation trials is given by

$$SE(ROP_{BOOT}) = \sqrt{\sum_{i=1}^{M} (\overline{ROP})^*(i,\cdot) - ROP_{BOOT})^2 / M(M-1)} . \tag{29}$$

The expression of standard error of (29) is the same as that given by Thombs and Schucany [30] and Bookbinder and Lordahl [32].

Realizations of Y_t and T are Both Available

Table I below shows the estimated ROP at 95% service level, probability coverage and the sample standard error for various estimation methods under the situation when (i) Y_t is assumed to follow an AR(2) process, and (ii) Y_t is AR(2) but was mistreated as *i.i.d.* *T* is assumed to follow a Poisson distribution. The ε_t for the AR (2) process is assumed to be normal with $\mu_\varepsilon = 0$ and $\sigma_\varepsilon = 5$. The parameter of the Poisson lead time is estimated from the sample realizations of n=20 lead times.

Table I

Among the five methods compared in Table I when Y_t is an AR(2) process, we observe that (i) the normal approximation underestimates the *ROP* and has the lowest probability coverage among all methods, (ii) the Pearson system overestimates *ROP* with large standard errors especially for the positively correlated demand, (iii) both the Pearson system and Charlier series estimate the *ROP* based on the first four moments of *L*, however, the Pearson system is more sensitive to the sampling variation, (iv) the Monte Carlo method is also sensitive to the sampling variation and has large standard errors, (v) the Charlier series and bootstrap procedure are preferable in terms of probability coverage and standard error among the five methods of estimating *ROP*.

When the correlations of the AR(2) demand process are ignored and the Y_t were treated as *i.i.d.*, all method underestimate substantially the *ROP* when the Y_t are positively correlated (see the first two cases of Table I), whereas all methods overestimate the *ROP* when the Y_t are negatively correlated (see the last case of Table I).

Sensitivity analysis for the probability coverage and standard errors over the lead time parameter and the lead time sample size are given. These are presented by Figures 1a and 1b for a comparison between (i) the Pearson system versus bootstrap procedure and (ii) an AR(2) of Y_t with

$(\phi_1, \phi_2) = (0.8, 0.1)$ versus an AR(2) of Y_t but was mistreated as *i.i.d.* The bootstrap and the Charlier series produce very similar result, therefore, only one of them is presented graphically.

FIGURES 1a and 1b

Figure 1a shows that when Y_t is modeled by an AR(2) process, the bootstrap procedure consistently provided good estimates for *ROP* with probability coverages that are close to 95% over different lead time parameter, whereas the Pearson system overestimated the *ROP* by about 2 percentage points. Figure 1b shows that bootstrap procedure which demonstrates consistently smaller standard error than the Pearson system in the correlated demands situation. It is also clear form Figure 1a that when the correlated demands were mistreated as *i.i.d*. then, both methods severely underestimate the service level. Although the mean μ_L is correctly estimated, the variance, the higher moments of *L* and *ROP* are estimated with error. The larger is the mean of the Poisson lead time, the worse is the estimated service level.

Figures 2a and 2b show the probability coverage and standard errors over the lead time sample size when Y_t is an AR(2) process with $(\phi_1, \phi_2) = (0.8, 0.1)$.

FIGURES 2a and 2b

Observing from Figures 2a and 2b we see that (i) an increase of lead time sample size has little improvement on the probability coverage, (ii) the Monte Carlo method and bootstrap procedure are comparable in probability coverage but the bootstrap procedure is more efficient with smaller standard errors, (iii) significant reduction in the standard errors is observed when the lead time sample size becomes increasingly large.

CONCLUDING REMARKS

The paper presents a general ARMA process for describing daily demand of an inventory, and the resulting decision on estimating the *ROP* for an inventory control situation. The first four moments of *L* obtained under this general ARMA demand structure provide a basis for estimating the ROP by several non-parametric methods.

The paper also addresses the issue on how to estimate the ROP based on the sample realizations in addition to the modeling aspect of the inventory demand process. This problem is usually not answered by the traditional inventory study in which the parameters in the process are assumed to be known and some types of sensitivity analysis for different values of the parameters are performed.

Since the choice of the appropriate lead time may affect the estimated ROP, a working guideline is therefore suggested. In light of the numerical investigation, it is worthwhile to note that the Charlier series is comparable to the Pearson system, but it has two advantages: (i) it is simpler to handle analytically as well as computationally; (ii) it yields a smaller standard error and is less sensitive to sampling variation than the Pearson system. The bootstrap procedure performs well in both the quality measurements on probability coverage and standard error. It shows high efficiency over the other methods in a small lead time sample situation. It is, however, the most computational intensive method among all the procedures.

When the demand is correlated as an AR(2) process but was mistreated as *i.i.d*„ all the methods show substantial estimation error for the Poisson lead time of which the variance-to-mean ratio is about one. Severe underestimation of *ROP* occurs when the positive autocorrelation of demand is ignored. Overestimation of *ROP* also occurs when the negative autocorrelation of demand is ignored. Substantial economic impact may result because of this misrepresentation of the demand structure.

Acknowledgement

The authors would like to thank Professor Sung K. Ahn and Stergios B. Fotoulos for their valuable comments and suggestions.

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TABLE I

Comparison of the Estimation results Between an AR(2) Demand and the Demand is Mistreated ad IID. Assuming a Poisson Lead Time with Parameter Estimated

			μ [a]								
(ϕ_1, ϕ_2) [b]	Model [c]	Method [d]	5			10			20		
			ROP	PC [e]	SE [f]	ROP	PC	SE	ROP	PC	SE
(0.8, 0.1)	AR2	P	281	96.9	2.9	500	97.9	5.4	824	96.6	7.8
		N	221	89.3	2.1	392	89.5	3.0	697	89.1	6.4
		C	253	94.4	2.3	445	94.9	3.2	777	94.6	6.8
		M	251	94.1	2.6	440	94.5	3.9	761	93.8	7.8
		B	252	94.2	2.2	438	94.4	3.2	762	93.8	6.9
	IID	P	188	80.2	2.3	326	76.9	3.6	571	72.0	6.9
		N	179	77.0	2.3	318	74.7	3.6	563	70.5	6.8
		C	189	80.5	2.3	327	77.1	3.7	571	72.2	6.9
		M	189	80.4	2.4	327	77.0	3.8	570	72.0	7.0
		B	188	80.2	2.2	325	76.6	3.6	569	71.7	6.9
(0.6, 0.3)	AR2	P	260	96.5	2.6	472	97.6	4.8	796	96.6	6.9
		N	214	90.3	1.9	374	89.4	2.8	687	90.3	6.8
		C	242	94.7	2.1	422	94.7	3.0	757	94.9	6.9
		M	239	94.3	2.5	414	94.1	3.0	750	94.6	7.6
		B	241	94.6	2.0	416	94.2	3.0	746	94.4	7.0
	IID	P	190	84.4	2.2	320	78.9	3.3	586	77.6	7.0
		N	182	81.6	2.1	312	76.9	3.3	578	76.3	6.9
		C	191	84.6	2.2	321	79.2	3.3	587	77.8	7.0
		M	191	84.6	2.4	315	77.5	3.7	586	77.6	7.2
		B	191	84.4	2.1	319	78.7	3.3	585	77.5	7.0
(-0.9, 0.0)	AR2	P	176	95.3	1.2	304	94.8	1.7	543	94.4	2.6
		N	165	92.7	1.2	294	93.2	1.7	533	93.2	2.6
		C	176	95.3	1.2	305	94.9	1.7	544	94.5	2.6
		M	173	94.9	1.7	304	94.8	2.7	536	93.5	3.5
		B	175	95.2	1.1	302	94.6	1.7	543	94.4	2.6
	IID	P	193	97.4	1.2	326	97.2	1.8	572	96.9	2.7
		N	184	96.5	1.2	318	96.4	1.8	564	96.3	2.7
		C	194	97.5	1.2	327	97.3	1.8	573	97.0	2.7
		M	192	97.3	2.0	324	97.0	2.7	568	96.7	3.2
		B	193	97.4	1.2	324	97.0	1.9	570	96.8	2.6

[a] : Parameter of the truncated Poisson lead time

[b] : True parameters of the AR(2) demand process $Y_t = a + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \varepsilon_t \;\; with \; \mu_Y = 20$

[c] : AR2: Use the true model, IID: the AR(2) demand is mistreated as *i.i.d.*

[d] : P: Pearson system, N: Normal approximation, C: Charlier Series, M: Monte Carlo simulation, B: Bootstrap.

[e] : PC= Cumulative probability of the estimated ROP

[f] : SE= Standard Error of the ROPs with 100 simulation runs

FIGURE 1a. Comparison of Probability Coverage Over Lead Time Parameter for the Pearson System Versus Bootstrap Procedure. AR(2) Versus IID Demands.

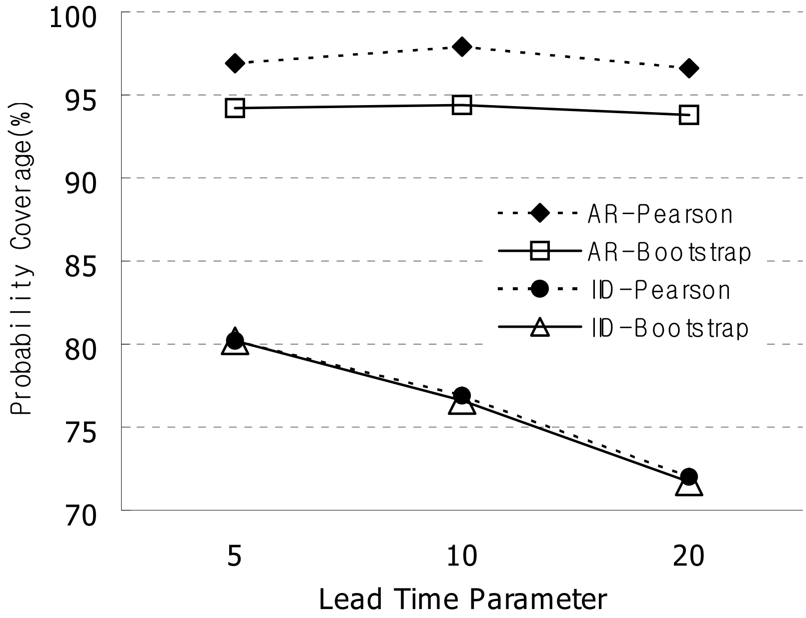

FIGURE 1b. Comparison of Standard Error Over Lead Time Parameter for the Pearson System Versus Bootstrap Procedure. AR(2) Versus IID Demands.

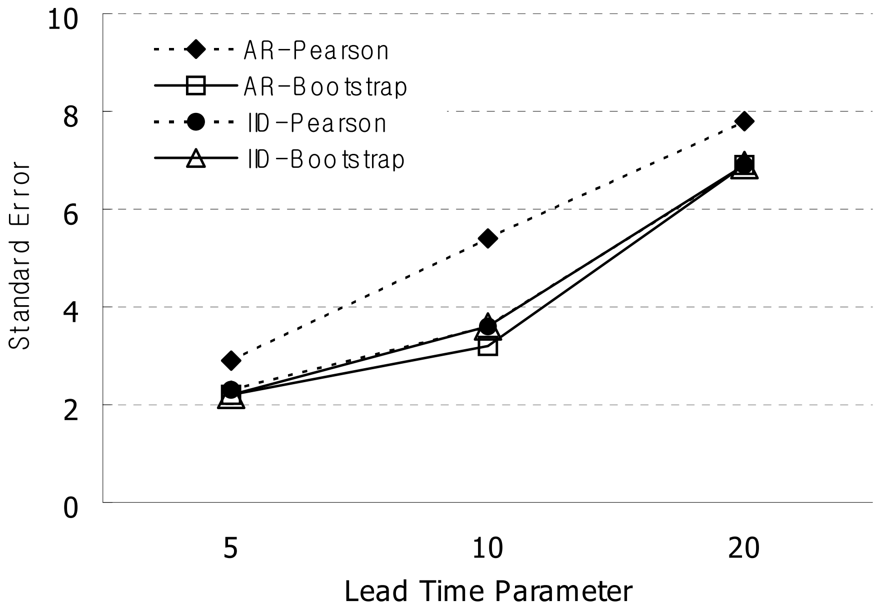

FIGURE 2a. Comparison of Probability Coverage Over Lead Time Sample Size for the Pearson System, Bootstrap and Monte Carlo Procedure. AR(2) Demands.

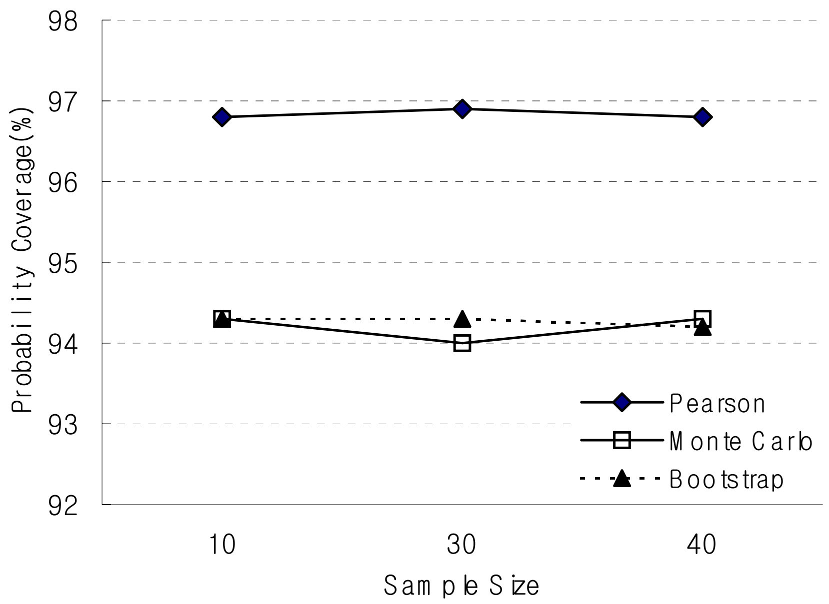

Figure 2b. Comparison of Standard Errors Over Lead Time Sample Size for the Pearson System, Bootstrap and Monte Carlo Procedure. AR(2) Demands**.**